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Chaotic Dynamical Systems Representation of Seismic Signals through Inverse Frobenius-Perron

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ABSTRACT

The historical seismic signals recorded for Surigao in the Caraga Region of the Philippines for 2011 to 2017 is modeled as a chaotic dynamical system whose ergodic density is assumed to obey a power-law distribution. This chaotic dynamical system is constructed by using the conjugate approach in finding solutions for the Inverse Frobenius-Perron Problem (IFPP). This new method of analysis determines the difference in distribution between the historical and generated seismic magnitudes. Results show a 76% similarity index between the simulated chaotic dynamics and the actual modified historical seismic data. The study concludes that a chaotic dynamical system can be used as a basis for

forecasting earthquake occurrences in this region of the Philippines.

Keywords: chaotic dynamical systems, Inverse Frobenius-Perron, seismic signals, ergodicity, equicontinuity

INTRODUCTION

Earthquake prediction is considered controversial apart from being a challenging task. Recent catastrophic earthquakes that went unpredicted such as those which occurred in Mexico (2018) and Japan (2010), are pieces of evidence of the challenges that confront seismic signal analysts worldwide. In many cases, statistical approaches have been utilized to model earthquake occurrences in time or space. A primary statistical model used to predict the temporal probability of earthquakes is the Poisson model, which, by definition, is a rare event model. As a result, the customarily used hypothesis should be largely associated with the prior judgment that earthquakes are rare but not abundant quantitative evidence or theoretical derivation. Wang et al. (2014) demonstrated that the Poisson hypothesis is valid for earthquakes of magnitude seven or greater but not for lower seismic intensities based on 55,000 events in Taiwan since 1900.On the other hand, Greenhough (2008) provided a method for estimating the uncertainties on the total number of events for a given period, still using the Poisson hypothesis. The main problem with employing stochastic processes to analyze seismic signal patterns is the significant amount of prediction uncertainty embedded in it rendering, many stochastic models unsuitable for practice.

On the other hand, if it is accepted that some forcing function exists that is responsible for the observed seismic signals. We can focus on the discovery of this underlying forcing function. In this context, dynamical systems analysis may replace stochastic modeling as the basis for earthquake prediction. A dynamical system is a sequence of variables $\{x_n : n \in Z^+\}$ such that:

$$x_n = \tau(x_n - 1) \tag{1}$$

When the output (x_n) behave like some random numbers, we say we have a chaotic dynamical system. Chaotic dynamical systems are suitable for the analysis of seismic signals because of their seemingly random behavior, although they are generated by a completely deterministic function τ (·).

The random numbers generated by (1) for large n obey an ergodic distribution obtained by the method of Frobenius and Perron (1978). In the case of the seismic signals, we demonstrate that the ergodic distribution is the power-law distribution:

$$f(x) = \alpha\, x^{-\lambda} \qquad (2)$$

To obtain the dynamical map $\tau(.)$, we use the technique of Inverse Frobenius Perronas cited by Nijun-Wei (2015). In this manner, we recover back the original seismic signals given the initial data point. Likewise, map (1) can forecast with greater accuracy the occurrence of a seismic signal of a given magnitude since the technique is based on a purely deterministic analysis.

The present paper is based on 7-year data imputed hourly in the Surigao, Caraga area of the Philippines. Only the 2011-2012 observations were utilized for illustration purposes.

THEORETICAL BASES

Chaotic Dynamics, Equicontinuity and Uniform Boundedness, Ergodicity, and Inverse Frobenius-Perron Operator

Definition 1.1. A stochastic process$\{X_t\colon t \in T\}$ is a sequence of random variables indexed by time T. If T is a finite or countably infinite set, then we have a discrete stochastic process. Otherwise, we have a continuous-time stochastic process (Padua, 2016).

There are many examples of stochastic processes that had been thoroughly studied, the most famous of which is the Brownian process. The values of $\{Xt\}$ may also be generated deterministically rather than by some random process. Define the map $f(.)$:

$$f\colon R \to R$$

such that:

$$X_{t+1} = f(X_t),\ t = 0,1,2, \ldots$$

The sequence $\{X_t\}$ also evolves T but is not defined by some random process and, hence, is not a stochastic process. This is called a *dynamical system*.

Definition 1.2. A *dynamical process* $\{X_t\colon t \in T\}$ is a sequence of deterministicvariables indexed by the time T such that there exists a continuous map $f\colon R \to R$ for which

$$X_{t+1} = f(X_t),\ t = 0,1,2,\ldots$$

Thus, if X_0 is known, then:

$$X_n = f^n(X0)$$

There exists a special class of deterministic dynamical systems for which the values behave like random numbers. This special class is known as *chaotic dynamical systems*.

Orbits and Periodic Points

The following definitions are taken from the lecture notes of Padua (2017).

Definition 1.3: Let $Xn+1 = f(Xn)$. The sequence $\{X_0, X_1 = f(X_0), X_2= f(X_1), ...\}$ is called the orbit of X_0, denoted by $\vartheta(X_0)$.

The first time that the iterates hit the initial point is of particular interest in dynamical systems.

Definition 1.4: A point X_p is a periodic point of $f(\cdot)$ if $f(X_p) = X_p$. The smallest positive integer n for which this is true is called the period of X_p. The period of X_p is defined as:

$$d = \inf \{n: f^n(X_p) = X_p\}$$

Definition 1.5. A periodic point X_p is an attracting fixed point if $|f'(X_p)| < 1$; a *repelling fixed point* if $|f'(Xp)| > 1$; is *an unstable fixed point* if $|f(Xp)| = 1$.

The presence of many periodic points which are either repelling or attracting gives the system the appearance of chaos or disorder. For a dynamical system to be *chaotic*, there should be infinitely many periodic points. Also, the following conditions should be satisfied:

1. The set of all periodic points S is close to every point in the system. That is,choose $x \in [D]$, then $\exists$ a periodic point $X_p \ni : |X - X_p| < \varepsilon \ \forall \ \varepsilon > 0$. This is called topological transitivity.

2. The intersection of any two orbits ϑ_1 and ϑ_2 starting from two initial points X_{01} and X_{02} are non-empty. That is $\vartheta_1(X01\ ) \cap \vartheta_2(X_{02}) \neq 0$.

3. The system is sensitive to initial conditions. This kind of sensitivity of this chaotic system is sometimes called the *butterfly effect* (Lorenz, 2001). Whichhighlight the possibility that small causes may have significant effects.

The Logistic Map

The logistic map is a model that is often used in population dynamics. Suppose that certain types of organisms are cultured in a confined space (see, for example, the fruit fly experiment of Robert May (1976)). Space has a maximum carrying capacity M. The number of organisms found at any given time T is expressed as a percentage of this maximum carrying capacity. Let $x(t)$ denote the percentage of individuals at time t. Then, the logistic growth model is given by:

$$x(t+1)= ax(t)[1-x(t)],\, t=0,1,2, \ldots$$

If 1 < a < 3.758.., then the long-term behavior of the population stabilizes at the value:

$$x(\infty)= \frac{a-1}{a}$$

Thus, if a = 2, the population stabilizes at x = 1/2; as the parameter $\boldsymbol{a}$ nears a =3, the values begin to fluctuate between two periodic points, one of which is at x = 2/3 . Further increase in the value of a forces the values to fluctuate between 4, 8, 16, periodic points until finally, at a = 4, an infinite number of periodic points are observed. It is worth noting that if we generated random numbers from an arcsine distribution on [0,1], then the graph of this sequence would be indistinguishable from a deterministic logistic map with a = 4. Chaotic systems, therefore, provide the platform for analyzing random systems from a deterministic point of view. Li and Yorke (1975) proved that if a dynamical system contains a periodic point of period 3, then the system is chaotic. While a point of period 3 implies the existence of one of period 5, the converse is false.

Frobenius-Perron Operator

Chaotic dynamical systems result in a trajectory, for any given starting point, that behaves as though we have a random-like trajectory. The random trajectory can be summarized by a probability distribution called its ergodic distribution and is obtained by the Frobenius-Perron method. More specifically, if the initial density function on the space I is $f(x)$, for a map $\tau: I \rightarrow I$, the density ϕ under the action of τ is $\phi = P\tau(f)$, where the operator $P\,\tau$ is called the Frobenius-Perron operator (FPO), or transfer operator,corresponding to τ.

Definition 1.6. Let (X, Σ, μ) be a σ – finite measure space, and let $S: X \to X$ be a nonsingular transformation; i.e., S is measurable and $\mu(S^{-1}(A)) = 0$ for all $A \in \Sigma$ such that$\mu(A)= 0$. In ergodic theory, the operator $P_S: L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ defined implicitly by

$$\int_A P_S f d\mu = \int_{S^{-1}(A)} f d\mu \quad (3)$$

is called the Frobenius-Perron operator associated with S.

Many problems in physical sciences are related to the problem of the existence of absolutely continuous invariant measures. It is obvious from (3) that for $f \in L^1(X, \Sigma, \mu)$, themeasure μ_f defined by

$$\mu_f(A) = \int_A f d\mu \qquad \text{for all } A \in \Sigma,$$

which is continuous with respect to μ, is invariant under S if and only if f is a fixed point of P_S. Here, the invariance of the measure μ_f (under S) means that $\mu(S^{-1}(A)) = \mu_f(A)$ for every measurable set A. Hence, the existence of an continuous invariant measure for a nonsingular transformation is equivalent to the fixed-point problem of the corresponding Frobenius-Perron operator.

Equicontinuity and Uniform Boundedness

Definition 1.7. O_k is uniformly equicontinuous on I if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon)$ such that $\forall t_1, t_2 \in I, |t_1 - t_2| < \delta$ then $|Ok(t_1) - Ok(t_2)| < \varepsilon$.

Definition 1.8. O_k is uniformly bounded on I if $\exists Q > 0$, such that $|O_k(t)| \leq Q, \forall t, k$.

Theorem 1.1 Arzela-Azcoli Theorem. If Ok is a sequence of uniformly equicontinuous function on I, uniformly bounded on $I = [c, d]$, then it has a subsequence of functions that converges uniformly on $[c, d]$.

Ergodic Density

Intuitively, the ergodic theory is concerned with taking certain (stationary) sequences and saying something about the convergence of the average of these sequences. If you have a function $f: R \to R$ and a (stationary) sequence $\{X_m\}_{m\geq 0}$ then under what conditions can you say

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m)$$

exists? From the strong law of large numbers (SLLN), we know that if the sequence is composed of independent and identically distributed (iid) random variables with $E|f(X_0)| < \infty$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) = \mu = EX_i .$$

the ergodic theorem is a sort of generalization of the SLLN. It states that if we impose some additional structure on $\{X_m\}_{m\geq 0}$, namely that the sequence is stationary and $E|f(X_0)| < \infty$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) \text{ exists.}$$

if the sequence has the additional property of being ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) = Ef(X_0).$$

Definition 1.7. A sequence $\{X_m\}_{m\geq 0}$ is said to be stationary if $P((X_0, X_1, \dots, Xm) \in A) = P((X_k, X_{k+1}, \dots, X_{k+m}) \in A)$ for all $m, k \geq 0$ and $A \in B^{m+1}$. We can say the distribution of X_n is the same as the shifted distribution for any shift value of k.

Definition 1.8. A map ϕ: $(\Omega, F) \to (\Omega, F)$ is said to be measure-preserving with respect to a probability measure P if $P(\phi A) = P(A)$ for all $A \in F$. That is the measure of the inverse image of a set is the same as the measure of the set.

Proof. To see why this theorem is true, let $B \in B^{n+1}$ and define $A = \{w: (X_0(w), X_1(w), \dots, X_n(w)) \in B\}$, then

$$\begin{aligned} P[(X_k(w), X_{k+1}(w), \dots, X_{k+n}(w)) \in B] &= P[(X(\phi^k w), X(\phi^{k+1} w), \dots, X(\phi^{k+n} w)) \in B] \\ &= P(\phi^k w \in A) \\ &= P(w \in A) \\ &= P(X_0, X_1, \dots, X_n) \in B \end{aligned}$$

where the second to the last equality is due to ϕ being measure preserving. Hence $X_n(w)$ is stationary. ■

Inverse Frobenius-Perron for Non-Symmetric Densities

The inverse Frobenius-Perron problem is associated with the Frobenius-Perron operator. Assume the space under consideration is the interval $I = [a, b]$ and points are distributed by a probability density function $f \in L$. That is, the probability of the initial point being in any measurable set $A \subset I$ is

$$Prob\{x \in A\} = \int_A f d\lambda,$$

where λ is the normalized Lebesgue measure on I. Let points being transformed by a map τ. After the transformation, the distribution over I would be different. Assume the new density is ϕ, then the probability function becomes

$$\int_A \phi d\lambda = Prob\{\tau(x) \in A\} = Prob\{x \in \tau^{-1}(A)\} = \int_{\tau^{-1}(A)} f d\lambda.$$

The existence of ϕ is given by the Radon-Nikodym Theorem. It is easy to see that ϕ is determined by τ and f. We let $P_\tau f$ denote ϕ. Then

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda. \qquad (4)$$

To generate a symmetric map with non-symmetric density $f(x)$, we shall use the transformation

$$\tau(x) = U^{-1}[1 - |U(x) - U(1-x)|] \quad . \qquad (5)$$

where $U(x) = u(x) + v(x)$, and $u(x) = 1 - u(1-x)$, $v(x) = v(1-x)$, $v(0) = 0$.

Claim: The invariant density f is $U'(x)$.
Proof: (see Nijun-Wei, 2015)

Define

$$U\ (x) = \frac{1}{-}\ {}^{x}[f(t) - f(1-t)]dt \qquad (6)$$

$$U^{+}_{-}(x) = \frac{1}{2}\int_0^x [f(t) + f(1-t)]dt. \qquad (7)$$

Notice that $u(x) = U_-(x)$, $v(x) = U_+(x)$ is a decomposition for U, that is, $u(x) = 1 - u(1 - x)$, $v(x) = v(1-x)$. Since,

$$U_-(x) = \frac{1}{2}\left(\int_0^x f(t)dt + \int_0^x f(1-t)dt\right)$$

$$= \frac{1}{2}\left(U(x) + \int_0^1 f(z)dz - \int_0^{1-x} f(z)dz\right)$$

$$= \frac{1}{2}(U\ (x) + 1 - U(1-x)),$$

we obtain the formula for the map

$$\tau(x) = U^{-1}[1 - |U(x) - U(1-x)|]. \qquad (8)$$

OBJECTIVE OF THE STUDY

This study intended to determine the difference in the distribution of the historical and generated seismic magnitudes greater or equal to a magnitude-4 by a new method of analysis using Dynamical systems of the 7-year data from Surigao, Caraga Region in the Philippines, which is highly at risk of earthquakes.

RESULTS AND DISCUSSION

The figure below shows the histogram of the historical data of the earthquake magnitudes in Surigao, Caraga Region for 2011-2017 with unequal spacings.

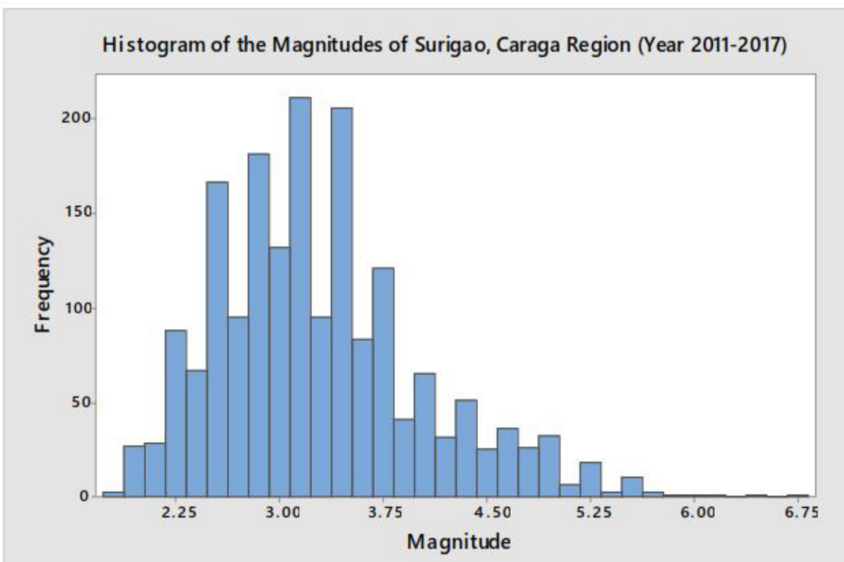

Figure 1. Histogram of the Magnitudes of Surigao, Caraga Region.

The unequal spacings of observations present a problem from a time series point of view. Likewise, the seismic magnitudes are not normalized which is a problem when we wish to use dynamical systems to model the observations. For normalization purposes, we utilize the established table from the US Geological Survey (Borradaile, 2003) which observed the largest earthquake magnitude at 9.2-moment magnitude scale. Normalization was done as follows:

$$\textit{normalized datum} = \frac{\textit{actual datum}}{0}$$

For missing observations, we imputed values based on a Bayesian non-informative prior. The lower endpoint is set at. 1 while the upper endpoint is fixed at the higher value of the observed data. The normalization process coupled with the Bayesian imputation technique resulted in modified data that have equal spacings at $h = 1\ hour$. The histogram of the modified data is illustrated below:

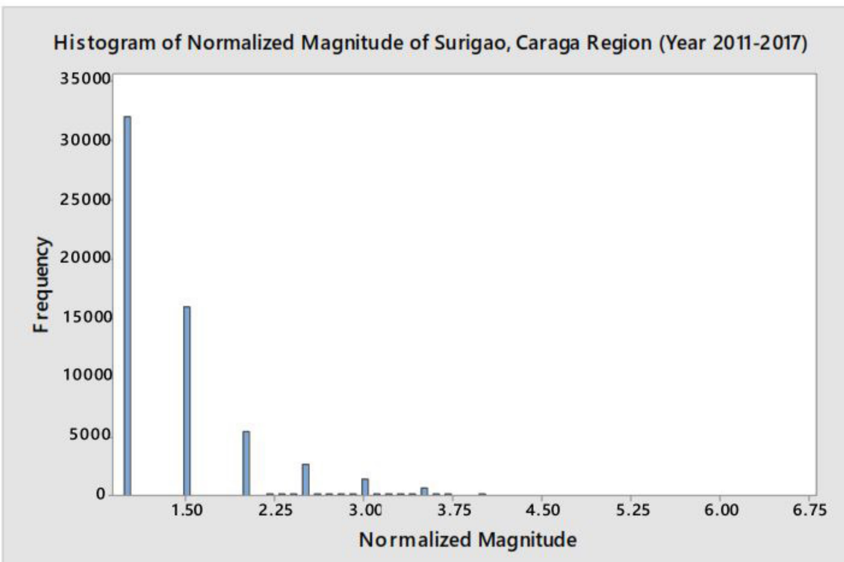

Figure 2. Histogram of Normalized Magnitude of Surigao, Caraga Region.

The histogram of the modified historical data suggests a power law ergodic distribution of the form:

$$f(x) \sim x^{-\lambda}$$

Result 1: If the seismic signals have an ergodic density with a power law form, then

$$f(x) = c[x^{1-\lambda} -. 1^{1-\lambda}], \qquad \text{where } c = \frac{1}{1 -. 1^{1-\lambda}}.$$

Proof.

Let

$$f(x) = cx^{-\lambda}, .1 \leq x \leq 1, \lambda \geq 1.$$

Then

$$F(x) = \int_{.1}^{1} cx^{-\lambda}dx = 1$$

$$c \int_{.1} x^{-\lambda}dx = 1$$

$$c\frac{x^{1-\lambda}}{1-\lambda}\Big|_{.1}^{1} = 1$$

$$\frac{c}{1-\lambda}[1 -. 1^{1-\lambda}] = 1$$

thus, $c = \frac{1-\lambda}{1-.1^{1-\lambda}}$ ■.

Result 2. The cumulative distribution function of the power law distribution describing the seismic signals is given by:

$$F(x) = c\,(\frac{x^{1-\lambda} -. 1^{1-\lambda}}{1-\lambda}), \qquad \text{where c is given in Result 1.}$$

Proof. The proof follows from Result 1 using the specific value of c found.

Result 3. The dynamical map generating the ergodic density for seismic signals is given by:

$$\tau = (\frac{[1 + |c(x^{1-\lambda} - (1-x)^{1-\lambda})|]}{c} +. 1^{1-\lambda})^{\frac{1}{1-\lambda}}$$

Proof.

By inversion formula as cited by Nijun-Wei (2015) let

$$U = c[x^{1-\lambda} -. 1^{1-\lambda}].$$

Hence,

$$\frac{U}{c} = x^{1-\lambda} -. 1^{1-\lambda}$$

$$\frac{U}{c} +. 1^{1-\lambda} = x^{1-\lambda}$$

$$x = (\frac{U}{c} +. 1^{1-\lambda})^{\frac{1}{1-\lambda}}.$$

which means that:

$$U^{-1}(x) = (\frac{x}{c} +. 1^{1-\lambda})^{\frac{1}{1-\lambda}}.$$

and

$$U(1-x) = c[(1-x)^{1-\lambda} -.\,1^{1-\lambda}].$$

Hence solving for τ:

$$\tau = U^{-1}[1 + |U(x) - U(1-x)|]$$

$$\tau = U^{-1}[1 + |c[x^{1-\lambda} -.\,1^{1-\lambda}] - c[(1-x)^{1-\lambda} -.\,1^{1-\lambda}]|]$$

$$\tau = U^{-1}[1 + |c(x^{1-\lambda} - (1-x)^{1-\lambda})|]$$

$$\tau = \left[\frac{\left\{1 + \left|c\left[\frac{x^{1-\lambda} - (1-x)^{1-\lambda}}{1-\lambda}\right]\right|\right\}(1-\lambda)}{c} +.\,1^{1-\lambda}\right]^{\frac{1}{1-\lambda}} \quad \blacksquare.$$

The maximum likelihood estimator of λ yielded a value close to 1.5 i.e. λ= 1.497757. Using this value λ, the chaotic dynamical function was iterated using the same starting value as the original data, namely, x = 0.34. The histogram of the resulting iterations is shown below: We note that the histogram of the modified historical data and the simulated chaotic dynamics data are almost identical.

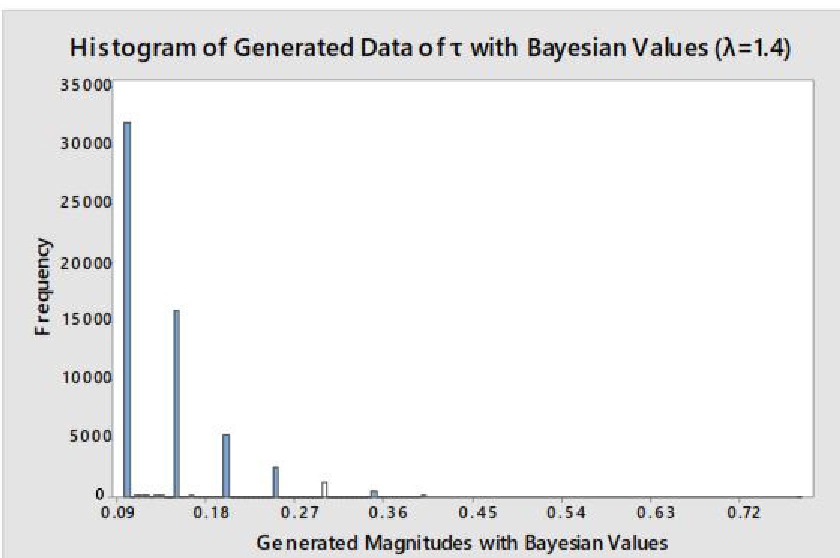

Figure 3. Histogram of Generated Data of $\boldsymbol{\tau}$ with Bayesian Values.

Moreover, the chaotic dynamics obeyed the same pattern as the original modified historical data with a correlation coefficient of r= 0.870 or R^2= 75.74%. It follows that the chaotic dynamics can be used for forecasting purposes with a mean squared error of MSE = 0.0012399 or a standard error of SE = 0.000392.

CONCLUSIONS

The above results show that the simulated chaotic dynamics and the actual modified historical seismic data of Surigao, Caraga Region has an 87% similarity index which was supported by the MSE/SE results. It is evident that the constructed chaotic dynamical map τ of this study definitely introduce a new kind of method for the analysis of seismic signals in this area.

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